TENSOR FUNCTIONS OF POLAR AND AXIAL VECTORS COMPATIBLE WITH TEXTURE SYMMETRY

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PMM Vol.28, № 4, 1964, pp. 653-663

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(Received April 4, 1964)

The general form of tensor functions of axial and polar vectors, compatiable with texture symmetry, is given. The sought functions (scalar; polar and axial vectors; symmetric, antisymmetric and general tensors of the second rank) are represented by means of a system of linearly independent vectors, which depend on products and contractions of the so called geometrical tensors. The coefficients of the expansion are arbitrary single valued scalar functions which depend on the mentioned quantities. The obtained expansions satisfy two important basic requirements: (1) each component of a tensor function is an entire rational function of the tensor argument, that is, a polynomial in its arguments, (2) the representation od each tensor function is unique, that is, the tensor function vanishes if and only if all of the coefficients in its expansion are identically zero, considered as polynomials in their arguments. A particular case of these functions is considered, that of the potential functions. The formulas given are the analogue, for vector argument and anisotropic medium, of the Hamilton-Cayley formulas, when these are considered as the general form of tensor functions of a tensor argument which are compatible with an isotropic medium.

In Section 1 the fundamental concepts and the formulation of the problem will be found while Sections 2 and 3 are devoted to the general method of solution employed. Sections 4 and 5 contain the general form of the tensor functions of a vector and an axial vector which are compatible with a texture symmetry. Potential functions are considered in Section 6. In Section 7 the tensor functions are applied in constructing invariant tensors . In Section 8 these results are related to the Hamilton-Cayley formula.

1. In many considerations of physics and mechanics of continuous media one has to employ relations of the form

$$\mathbf{T} = F\,(\mathbf{A}_{(1)}, \ldots, \mathbf{A}_{(m)}) \tag{1.1}$$

where the tensors $\mathbf{T}$ and $\mathbf{A}_{(\mu)}$ $(\mu = 1,\ldots,m)$ are so called field tensors which describe certain physical fields.

The material tensors $\mathbf{D}_{(\sigma)}$ $(\sigma = 1,\ldots,s)$, which determine the characteristic properties of the medium, usually appear explicitly in (1.1)

$$\mathbf{T} = F\,(\mathbf{D}_{(1)}, \ldots, \mathbf{D}_{(s)};\ \mathbf{A}_{(1)}, \ldots, \mathbf{A}_{(m)}) \tag{1.2}$$

The relations (1.1) and (1.2) depict the tensor field $\mathbf{T}$ which is produced at a certain point of the medium as a consequence of the action at this point of the tensor fields $\mathbf{A}_{(\mu)}$.

Specific cases of the relations (1.2) are, for example, the well-known Hooke's law equations for anisotropic media and their extensions to the case of piezo-electric and elastic-viscous media. As a matter of fact, these equations represent only the first terms of the Taylor expansions of the components of the stress tensor in terms of the components of the deformation tensor, the velocity of deformation tensor, and the electric field vector. In order to take into account the higher order terms in the expansions, one has to introduce material tensors $\mathbf{D}_{(\sigma)}$ of higher rank. For example, the expansion of the component T^{ij} of a second order rank tensor $\mathbf{T}$ in terms of the components A_k of the vector $\mathbf{A}$ has the form

$$T^{ij} = D^{ij} + D^{ij}_{..k}A^k + D^{ij}_{..kl}A^kA^l + \dots \tag{1.3}$$

The tensors $\mathbf{D}_{(\sigma)}$, which describe the properties of the contunuous medium, must be invariant with respect to the group of point symmetry G of the medium (field tensors, of course, are not required to satisfy this condition). The consideration of the quadratic, cubic, and higher order approximations requires the construction of the corresponding higher rank tensors which are invariant with respect to the point group. Solution of this problem, although elementary, is extremely complicated.

Nevertheless, the expansion (1.3) is not completely satisfactory, not only because of the technical difficulties already mentioned, but also because, as a matter of fact, it represents only polynomial dependence of the components of the tensor $\mathbf{T}$ on the components of the tensor $\mathbf{A}_{(\mu)}$. Nominally, the series (1.3) represents any analytic function $\mathbf{T} = F(\mathbf{A}_{(1)}, \dots, \mathbf{A}_{(m)})$, but practically the series is always cut off after a few terms and a satisfactory accuracy is obtained in a certain vicinity of the expansion center. Besides, in mechanics of continuous media one employs, sometimes, models which exibit a nonanalytic dependence of some tensors on others.

In the mechanics of isotropic media, already long ago one had considered an arbitrary functional dependence between tensors which are compatible with such a medium. The theory of these isotropic tensor functions is described, for example, in [1]. They are widely used in the nonlinear theory of elasticity [2] and of an elastic-viscous [3] isotropic continuous medium. It seems natural to seek to extend the theory of tensor functions to an anisotropic continuous medium. In this paper this problem is solved when the point group of symmetry of the anisotropic medium is one of the limit point groups of symmetry of Pierre Curie (*), the argument of the function is a polar or axial vector, while the function itself is a polar or axial vector, or a symmetric or general tensor of the second rank.

In order to formulate more precisely the problem, one needs the notions of internal and external symmetries of tensors. The group of internal symmetry of a tensor [5] is, in the general case, an antisymmetric group [6].

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*) Anisotropic media with this kind of symmetry are often called textures (see, for example, [4]).

Its operations are permutations of indices which leave invariant all components of a tensor, and its antioperations are the permutations which reverse the sign of each component. The group of internal symmetry of a tensor may be represented by the transformation τ of the orthogonal group which transforms its own components. If a tensor of rank r does not possess internal symmetry, then its components are represented by the rth power of a vector representation V of the orthogonal group: $\tau = V^r$. If the tensor possesses a nontrivial internal symmetry, then τ is a certain symmetrization of an rth power of a vector representation V. The symbol of this representation will be used, following Jahn [7], to denote the internal symmetry of a tensor. Thus, the symmetry of a tensor of rank r, with resect to every index, will be designated by means of the symbol of an rth symmetric power of a vector representation $[V^r]$ and the symmetry and antisymmetry with resect to s indices, where $s < r$, will be denoted respectively by $[V^s]V^{r-s}$ and $\{V^s\}V^{r-s}$, etc.

The group of exterior symmetry [8] of a tensor (*) is the maximal point group G which leaves invariant all the components of a tensor; if $c_i^{i'}(g)$ is a transformation belonging to the group G of exterior symmertry of a tensor $A^{i_1 \dots i_r}$, and only in this case

$$\left(c_{i_1}^{i_1'}(g)\dots c_{i_r}^{i_r'}(g) - \delta_{i_1}^{i_1'}\dots\delta_{i_r}^{i'_r}\right)A^{i_1\dots i_r} = 0 \qquad (g \in G) \tag{1.4}$$

Employing the terminology just introduced, let us formulate the problem before us. Suppose that the components of a tensor **T** of rank r and internal symmetry τ are functions of the components of a tensor **A** of rank s, internal symmetry α and exterior symmetry G_A

$$T^{i_1\dots i_r} = F^{i_1\dots i_r}(A^{j_1\dots j_s}) \tag{1.5}$$

Since **T** and **A** are tensors, then, under an arbitrary transformation of coordinates $a^i{}_i$, the transformed components of the tensors **T** and **A** must satisfy Equation

$$a^{i_1}_{i'_1}\dots a^{i_r}_{i'_r} T^{i_1'\dots i_r'} = F^{i_1\dots i_r}(a^{j_1}_{j_1'}\dots a^{j_s}_{j'_s} A^{j'_1\dots j_s'}) \tag{1.6}$$

If $a^i_{i'} = c^i_{i'}(g)$, $g \in G$ is one of the point transformations of the group G, then Equations (1.5) and (1.6) must be equivalent. The problem consists in finding the most general form of the functions G, G_A, which are such that, for given $F^{i_1\dots i_r}$ and α, the equations (1.5) and (1.6) are indeed equivalent.

2. The components of the tensor **T** must, obviously, remain invariant under all transformations which belong to the point (**) group Γ which is the intersection of the point group G of symmetry of the medium and the group G_A of external symmetry of the tensor **A**. The set of all tensors which fulfill this requirement and possess the internal symmetry τ, constitute a linear space, whose dimension, as follows from the theory of group representations [9], equals

$$n = \frac{1}{N(\Gamma)}\sum_{g\in\Gamma}\chi_\tau(g) \tag{2.1}$$

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*) In [8], where this concept was firts introduced, the term "tensor symmetry" was employed.

) The group Γ may coincide with the group G_T of external symmetry of the tensor **T, but, in general, it is only a subgroup.

Here $N(\Gamma)$ denotes the order of the group Γ and $\chi_\tau(g)$ is the trace of the matrix corresponding to the element g of the group Γ under the representation τ of this group. If Γ is one of the limit point groups, then the summation is understood to mean integration over the group.

The tensor $\mathbf{T}$, as in [1, 4, 5 and 10] will be sought in the form

$$\mathbf{T} = \sum_{\nu=1}^{n} f_{(\nu)} \mathbf{Q}_{(\nu)} \tag{2.2}$$

where $\mathbf{Q}_{(\nu)}$ $(\nu = 1, \ldots n)$ are linearly independent tensors of internal symmetry τ , whose components are invariant with respect to the group Γ . Since the tensors $\mathbf{Q}_{(\nu)}$ constitute a basis for the linear space of tensors $\mathbf{T}$, it is true that any tensor having the required properties may be represented in the form (2.2). The coefficients $f_{(\nu)}$, of this expansion, as well as the components of the tensors $\mathbf{Q}_{(\nu)}$, are certain functions of the components of the tensor $\mathbf{A}$. In the calculations we shall employ the so called geometrical tensors $\mathbf{K}_{(\pi)}$ $(\pi = 1, \ldots p)$ (introduced by Sedov and Lokhin in [4 and 11]), which is a set of p tensors with constant components, determining uniquely the point group G , the intersection of the external symmetry groups of these tensors.

Obviously, all the scalars obtained by means of multiplications and contractions from the geometrical tensors $\mathbf{K}_{(\pi)}$, of the group G , and the tensor $\mathbf{A}$, are, themselves, invariants of the tensor $\mathbf{A}$ with respect to the group G , and, desides, are integral rational invariants. As id known [12 and 13], all those invariants may be obtained by means of multiplications and linear combinations out of a certain finite set $\varphi_1, \ldots, \varphi_k$, the so called integral rational basis. The number k may coincide with the number

$$k_0 = \frac{1}{N(\Gamma)} \sum_{g \in \Gamma} \chi_\alpha(g) \tag{2.3}$$

of functionally independent invariants, but it may also exceed it.

We shall call a representation $f(\varphi_1, \ldots, \varphi_k)$ unique, whenever f , considered as a function of the components $A^{j_1 \cdots j_s}$. is identically zero; it follows that it must also equal zero when considered as a function of the invariants φ_k , thought of as independent arguments. In the case $k = k_0$ each invariant f may be uniquely written in the form $f(\varphi_1, \ldots, \varphi_k)$. If $k > k_0$, let us isolate, from the total number of invariants, k_0 which are functionally independent: $\varphi_1, \ldots, \varphi_{k_0}$ (let us call them "principal"), and let the remaining ("complementary") invariants be denoted by

$$\psi_1, \ldots, \psi_l \ (l = k - k_0).$$

Then it may be proved that [14] an arbitrary invariant Φ may be uniquely written in the form

$$\Phi = f_0 + \sum_{\lambda=1}^{l} \psi_\lambda f_\lambda + \sum_{\lambda=1}^{\mu} \sum_{\mu=1}^{l} \psi_\lambda \psi_\mu f_{\lambda\mu} + \cdots \tag{2.4}$$

where $f_0, f_\lambda, f_{\lambda\mu}, \cdots$ are, either arbitrary functions of the principal invariants or else are identically zero, and where, besides, beginning with a certain index t , all $f_{\lambda_1 \ldots \lambda_t}$, $f_{\lambda_1 \ldots \lambda_t \lambda_{t+1}}, \cdots$ are equal to zero (and thus the sum (2.4) has only a finite number of terms). Further, in the concrete cases of

Sections 4 to 6, we will have $k = k_0$.

In many papers [14 to 16] it is supposed that f and Φ are polynomials in the invariants $\varphi_\varkappa$ and the components $A_{j_1 \ldots j_s}$. This restriction, as shown in [17], may be easily removed. For arbitrary functions, on the other hand, the concept of unique representation, which is very easy for polynomials, becomes meaningless. For this reason, the functions f will be supposed, in the sequel, to be piecewise analytic, for the needs of practice this class is wide enough, and the concept of unique representation does not involve any doubts in this case.

From the geometric tensors $\mathbf{K}_{(\pi)}$ and the tensor $\mathbf{A}$, by means of multiplications and contractions, we may obtain also the set of tensors of internal symmetry τ . All of these, obviously, are invariant with respect to the group Γ . Formula (2.1) shows that from these one may choose (in a variety of ways) exactly n linearly independent tensors. These may be used as a basis in the expansion (2.2) of the tensors $\mathbf{Q}_{(\nu)}$.

Tensor functions of the form

$$T^{i_1 \ldots i_r} = \sum_{\nu=1}^{n} f_{(\nu)} (\varphi_1, \ldots, \varphi_k)\, Q_{(\nu)}^{i_1 \ldots i_r} \qquad \text{for} \quad k = k_0 \tag{2.5}$$

or

$$T^{i_1 \ldots i_r} = \sum_{\nu=1}^{n} \Phi_{(\nu)} Q_{(\nu)}^{i_1 \ldots i_r} \qquad \text{for} \quad k > k_0 \tag{2.6}$$

obviously satisfy the requirements. Indeed, since the functions $f_{(\nu)}$ and $\Phi_{(\nu)}$, and also the geometric tensors $\mathbf{K}_{(n)}$ are invariants of the transformations of the group G, Equations (1.6), for $a_{i'}^{i} = c_{i'}^{i}(g)$, $g \in G$ are simply a linear combination of Equations (1.5), hence the passage from the system (1.5) to the system (1.6) is reversible. Consequently, the system (1.5) and (1.6) are equivalent.

3. Since, in applications, one often restricts attention to entire rational tensor functions, it is natural to restrict further the expansions (2.5) and (2.6). Then, let us require further: (1) that $f_{(\nu)}(\varphi_1, \ldots, \varphi_k)$ be polynomials in their arguments whenever $T^{i_1 \ldots i_r}$ are polynomials in the $A^{j_1 \ldots j_s}$, (2) that the expansions (2.5) and (2.6) be unique, that is, that $\mathbf{T} = 0$ if and only if all $f_{(\nu)} \equiv 0$.

The method followed here, in view of the linear independence of the tensors $\mathbf{Q}_{(\nu)}$, and the uniqueness of the basis functions $f_{(\nu)}$ – certainly satisfies the second requirement, but the first requirement remains to be verified. One may employ another method suggested by Rivlin [15 and 16], then the first requirement is clearly satisfied, but it remains to verify that the second one is actually satisfied.

Let us indicate how to verify that the second requirement is fulfilled. Let us expand the tensor $\mathbf{T}$ in powers of the tensor $\mathbf{A}$

$$\mathbf{T} = \sum_{q} \mathbf{T}_{(q)} \tag{3.1}$$

where the components of the tensor $\mathbf{T}_{(q)}$ are homogeneous polynomials of degree q with respect to the components of $\mathbf{A}$. The number of linearly independent (*) components of the tensor $\mathbf{T}_{(q)}$ is

$$m_q = \frac{1}{N(G)} \sum_{g \in G} \chi_\tau(g)\, [\chi_\alpha{}^q](g) \tag{3.2}$$

and, in particular

$$m_0 = \frac{1}{N(G)} \sum_{g \in G} \chi_\tau(g) \tag{3.3}$$

Here $[\chi_\alpha{}^q](g)$ is the trace of the matrix of the representation $[\alpha^q]$ which corresponds to the element g of the group G.

On the other hand, the number m_q^* of terms of degree q, with respect to the components of $\mathbf{A}$, in the sum $\Sigma f_{(\nu)} Q_{(\nu)}^{i_1 \dots i_r}$ can be easily calculated by the methods of combinatorial analysis [18].

If, as it happens in this case, $k = k_0$, then the number of terms of degree q in each of the polynomials $f_{(\nu)}$ equals the coefficient of t^q in the formal expansion

$$\prod_{\varkappa=1}^{k} (1 - t^{b(\varkappa)})^{-1}$$

where $b(\varkappa)$ is the degree of the principal invariant $\varphi_\varkappa$ with respect to $\mathbf{A}$. From the definition of the tensor $\mathbf{Q}_{(\nu)}$ it follows that its components are homogeneous polynomials of the same degree $c(\nu)$ with respect to $\mathbf{A}$. Therefore, m_q^* equals the coefficient of t^q in the formal expansion

$$F^*(t) = \sum_{\nu=1}^{n} t^{c(\nu)} \prod_{\varkappa=1}^{k} (1 - t^{b(\varkappa)})^{-1} \tag{3.4}$$

If $k > k_0$, then $F^*(t)$ can be obtained almost as easily [14].

The equality of the numbers m_q and m_q^*, for every q, is the criterion of fulfillment of the first requirement.

All reducible formulas in Sections 4 and 5 are verified for linear independence of the tensors $\mathbf{Q}_{(\nu)}$ and for equality of numbers m_q and m_q^* at all values of q.

Formula (3.2), however, is not convenient for calculation of m_q. Let us expand the representation $\tau \times [\alpha^q]$ of the orthogonal group in irreducible representations.

If q is odd then in the vector representation $(\alpha = V = D_1{}^u)$ we have

$$[\alpha^q] = D_1{}^u + D_3{}^u + \dots + D_q^u$$

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*) Here the coefficients of the linear relations are numbers, and not functions of the components of $\mathbf{A}$, as was the case in (2.5) and (2.6).

and, for the axial-vector representation $(\alpha = \{V^2\} = (D^g{}_1)$ we have

$$[\alpha^q] = D_1^g + D_3^g + \ldots + D_q^g$$

If q is even, then in both cases we have that

$$[\alpha^q] = D_0^g + D_2^g + \ldots + D_q^g$$

where D_j^g and D_j^u are conventional notations for the even and odd irreducible representations of weight j of the orthogonal group (see [19]). The expansion of τ in terms of irreducible representations of the orthogonal group, and the subsequent multiplying of the irreducible representations may be carried out in an elementary way [9]. The number m_q equals the number of unique representations of $\tau \times [\alpha^q]$ in irreducible representations of the group G. Hence, if G is the orthogonal group, then m_q is the number of D_0^g representations in the expansion of $\tau \times [\alpha^q]$ in irreducible representations of the orthogonal group, while if G is the group of rotations, then m_q is the sum of the D_0^g and D_0^u representations in this expansion. The formulas for the reduction of the irreducible representations of the orthogonal group to the representations of its subgroups [7], enable one to compute the number of unique representations of each group in the expansion of $\tau \times [\alpha^q]$, that is, to determine m_q for all the remaining groups.

4. The sets of the geometrical tensors $\mathbf{K}_{(\pi)}$, which determine the limit point groups, were given by Sedov and Lokhin [4 and 11]; here these sets will be in a slightly modified form, more convenient for our application.

Let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be unit vectors of a Cartesian system of coordinates, with $\mathbf{e}_3$ directed along the principal axis. Let their components in an arbitrary coordinate system be denoted by ξ^i, η^i and ζ^i, respectively. For brevity, we shall employ the notation $\xi^i\xi^j\ldots\xi^k \equiv \xi^{ij\ldots k}$ and the analogous ones $\eta^{ij\ldots k}$ and $\zeta^{ij\ldots k}$.

The geometrical tensors for all the limit groups (*), in this notation, are given by

group $\infty/\infty\cdot m$	$g^{ik} \equiv \xi^{ik} + \eta^{ik} + \zeta^{ik}$	
group ∞/∞	g^{ik}, $E^{ijk} \equiv 6\xi^{[i}\eta^j\zeta^{k]}$	
group $m\cdot\infty : m$	ζ^{ik}, $\gamma^{ik} \equiv g^{ik} - \zeta^{ik}$	
group $\infty : 2$	ζ^{ik}, γ^{ik}, $\zeta^i\Omega^{jk}$	$(\Omega^{jk} \equiv 2\xi^{[j}\eta^{k]})$
group $\infty\cdot m$	ζ^i, γ^{ik}	
group $\infty : m$	ζ^{ik}, γ^{ik}, Ω^{ik}	
group ∞	ζ^i, γ^{ik}, Ω^{ik}	

We shall next write down the basis invariants φ_K of a vector $\mathbf{B}$, scalar functions $f(\mathbf{B})$, vector functions $\mathbf{V}(\mathbf{B})$, antisymmetric tensor functions of the second rank $\mathbf{W}(\mathbf{B})$, and symmetric tensor functions of the second rank $\mathbf{S}(\mathbf{B})$, compatible with the symmetries of an anisotropic medium, for each of the seven limit point groups.

*) The groups are designated as was done by Shubnikov [6].

Groups $\infty/\infty\cdot m$ and ∞/∞

$$\varphi = g_{\alpha\beta}B^\alpha B^\beta, \quad f = f(\varphi), \quad V^i = fB^i, \quad S^{ik} = f_1 g^{ik} + f_2 B^i B^k$$
$$W^{ik} = 0 \quad \text{for } \infty/\infty\cdot m, \quad W^{ik} = fE^{ik}_{\cdot\cdot\alpha}B^\alpha \quad \text{for } \infty/\infty \tag{4.1}$$

Group $m\cdot\infty:m$

$$\varphi_1 = \zeta_{\alpha\beta}B^\alpha B^\beta, \quad \varphi_2 = \gamma_{\alpha\beta}B^\alpha B^\beta, \quad f = f(\varphi_1, \varphi_2)$$
$$V^i = (f_1\zeta^i_\alpha + f_2\gamma^i_\alpha)B^\alpha; \quad W^{ik} = f\zeta^{[i}_\alpha\gamma^{k]}_\beta B^\alpha B^\beta$$
$$S^{ik} = f_1\zeta^{ik} + f_2\gamma^{ik} + (f_3\gamma^i_\alpha\gamma^k_\beta + f_4\zeta^{(i}_\alpha\gamma^{k)}_\beta)B^\alpha B^\beta \tag{4.2}$$

Group $\infty:2$

$$\varphi_1 = \zeta_{\alpha\beta}B^\alpha B^\beta, \quad \varphi_2 = \gamma_{\alpha\beta}B^\alpha B^\beta, \quad f = f(\varphi_1, \varphi_2)$$
$$V^i = (f_1\zeta^i_\alpha + f_2\gamma^i_\alpha)B^\alpha + f_3\zeta_\alpha\Omega^i_{\cdot\beta}B^\alpha B^\beta$$
$$W^{ik} = (f_1\zeta_\alpha\Omega^{ik} + f_2\zeta^{[i}\Omega^{k]}_{\cdot\alpha})B^\alpha + f_3\zeta^{[i}_\alpha\gamma^{k]}_\beta B^\alpha B^\beta$$
$$S^{ik}=f_1\zeta^{ik}+f_2\gamma^{ik}+f_3\zeta^{(i}\Omega^{k)}_{\cdot\alpha}B^\alpha+(f_4\gamma^i_\alpha\gamma^k_\beta+f_5\zeta^{(i}_\alpha\gamma^{k)}_\beta)B^\alpha B^\beta+f_6\zeta_\alpha\Omega^{(i}_{\cdot\beta}\gamma^{k)}_\lambda B^\alpha B^\beta B^\lambda \tag{4.3}$$

Group $\infty\cdot m$

$$\varphi_1 = \zeta_\alpha B^\alpha, \quad \varphi_2 = \gamma_{\alpha\beta}B^\alpha B^\beta, \quad f = f(\varphi_1, \varphi_2)$$
$$V^i = f_1\zeta^i + f_2\gamma^i_\alpha B^\alpha; \quad W^{ik} = f\zeta^{[i}\gamma^{k]}_\alpha B^\alpha$$
$$S^{ik} = f_1\zeta^{ik} + f_2\gamma^{ik} + f_3\zeta^{(i}\gamma^{k)}_\alpha B^\alpha + f_4\gamma^i_\alpha\gamma^k_\beta B^\alpha B^\beta \tag{4.4}$$

Group $\infty:m$

$$\varphi_1 = \zeta_{\alpha\beta}B^\alpha B^\beta, \quad \varphi_2 = \gamma_{\alpha\beta}B^\alpha B^\beta, \quad f = f(\varphi_1, \varphi_2)$$
$$V^i = (f_1\zeta^i_\alpha + f_2\gamma^i_\alpha + f_3\Omega^i_{\cdot\alpha})B^\alpha, \quad W^{ik} = f_1\Omega^{ik} + (f_2\zeta^{[i}_\alpha\gamma^{k]}_\beta + f_3\zeta^{[i}_\alpha\Omega^{k]}_{\cdot\beta})B^\alpha B^\beta$$
$$S^{ik} = f_1\zeta^{ik} + f_2\gamma^{ik} + (f_3\gamma^i_\alpha\gamma^k_\beta + f_4\zeta^{(i}_\alpha\gamma^{k)}_\beta + f_5\zeta^{(i}_\alpha\Omega^{k)}_{\cdot\beta} + f_6\gamma^{(i}_\alpha\Omega^{k)}_{\cdot\beta})B^\alpha B^\beta \tag{4.5}$$

Group ∞

$$\varphi_1 = \zeta_\alpha B^\alpha, \quad \varphi_2 = \gamma_{\alpha\beta}B^\alpha B^\beta, \quad f = f(\varphi_1, \varphi_2)$$
$$V^i = f_1\zeta^i + (f_2\gamma^i_\alpha + f_3\Omega^i_{\cdot\alpha})B^\alpha, \quad W^{ik} = f_1\Omega^{ik} + (f_2\zeta^{[i}\gamma^{k]}_\alpha + f_3\zeta^{[i}\Omega^{k]}_{\cdot\alpha})B^\alpha$$
$$S^{ik} = f_1\zeta^{ik} + f_2\gamma^{ik} + (f_3\zeta^{(i}\gamma^{k)}_\alpha + f_4\zeta^{(i}\Omega^{k)}_{\cdot\alpha})B^\alpha + (f_5\gamma^i_\alpha\gamma^k_\beta + f_6\gamma^{(i}_\alpha\Omega^{k)}_{\cdot\beta})B^\alpha B^\beta \tag{4.6}$$

5. The axial vector **H** may be written (as **W** was in the preceding Section) in terms of its antisymmetric dual tensor of the second rank H^{ik}.

Groups $\infty/\infty\cdot m$ and ∞/∞

$$\varphi = g_{\alpha\lambda}g_{\beta\mu}H^{\alpha\beta}H^{\lambda\mu}; \quad f = f(\varphi)$$
$$W^{ik} = fH^{ik}; \quad S^{ik} = f_1 g^{ik} + f_2 g_{\alpha\beta}H^{i\alpha}H^{k\beta}$$
$$V^i = 0 \quad \text{for } \infty/\infty\cdot m, \quad V^i = fE^i_{\cdot\alpha\beta}H^{\alpha\beta} \quad \text{for } \infty/\infty \tag{5.1}$$

Groups $m\cdot\infty : m,\ \infty : 2,\ \infty\cdot m$

$$\varphi_1 = \gamma_{\alpha\lambda}\gamma_{\beta\mu}H^{\alpha\beta}H^{\lambda\mu}, \qquad \varphi_2 = \gamma_{\alpha\lambda}\zeta_{\beta\mu}H^{\alpha\beta}H^{\lambda\mu}, \qquad f = f(\varphi_1, \varphi_2)$$

$$W^{ik} = (f_1\zeta_\alpha^{[i}\gamma_\beta^{k]} + f_2\gamma_\alpha^i\gamma_\beta^k)\, H^{\alpha\beta} + f_3\zeta_\alpha^{[i}\gamma_\lambda^{k]}\gamma_{\beta\mu}H^{\alpha\beta}H^{\lambda\mu}$$

$$S^{ik} = f_1\zeta^{ik} + f_2\gamma^{ik} + f_3\zeta_\alpha^{(i}\gamma_\beta^{k)}H^{\alpha\beta} + (f_4\gamma_\alpha^i\gamma_\lambda^k\zeta_{\beta\mu} + f_5\zeta_\alpha^{(i}\gamma_\lambda^{k)}\gamma_{\beta\mu})\, H^{\alpha\beta}H^{\lambda\mu} + \\ + f_6\gamma_\alpha^{(i}\gamma_\lambda^{k)}\gamma_{\beta\rho}\gamma_{\mu\sigma}H^{\alpha\beta}H^{\lambda\mu}H^{\rho\sigma}$$

$$V^i = 0 \quad \text{for } m\cdot\infty : m$$

$$V^i = (f_1\zeta^i\Omega_{\alpha\beta} + f_2\zeta_\alpha\Omega^i_{\cdot\beta})\, H^{\alpha\beta} + f_3\zeta_\alpha\Omega_{\beta\lambda}\gamma_\mu^i H^{\alpha\beta}H^{\lambda\mu} \quad \text{for } \infty : 2$$

$$V^i = f_1\zeta^i + f_2\zeta_\alpha\gamma_\beta^i H^{\alpha\beta} + f_3\zeta_\alpha\gamma_\lambda^i\gamma_{\beta\mu}H^{\alpha\beta}H^{\lambda\mu} \quad \text{for } \infty\cdot m \tag{5.2}$$

Groups $\infty : m$ and ∞

$$\varphi_1 = \Omega_{\alpha\beta}H^{\alpha\beta}, \qquad \varphi_2 = \gamma_{\alpha\lambda}\zeta_{\beta\mu}H^{\alpha\beta}H^{\lambda\mu}$$

$$f = f(\varphi_1, \varphi_2), \qquad W^{ik} = f_1\Omega^{ik} + (f_2\zeta_\alpha^{[i}\gamma_\beta^{k]} + f_3\zeta_\alpha^{[i}\Omega^{k]}_{\cdot\beta})\, H^{\alpha\beta}$$

$$S^{ik} = f_1\zeta^{ik} + f_2\gamma^{ik} + (f_3\zeta_\alpha^{(i}\Omega^{k)}_{\cdot\beta} + f_4\zeta_\alpha^{(i}\gamma_\beta^{k)})\, H^{\alpha\beta} + (f_5\gamma_\alpha^i\gamma_\lambda^k\zeta_{\beta\mu} + \\ + f_6\gamma_\alpha^{(i}\Omega^{k)}_{\cdot\lambda}\zeta_{\beta\mu})\, H^{\alpha\beta}H^{\lambda\mu}$$

$$V^i = 0 \quad \text{for } \infty : m, \qquad V^i = f_1\zeta^i + (f_2\gamma_\alpha^i\zeta_\beta + f_3\Omega^i_{\cdot\alpha}\zeta_\beta)\, H^{\alpha\beta} \quad \text{for } \infty \tag{5.3}$$

6. In the applications to material tensors one often encounters additional restrictions of symmetry concerning the indices. If, for example, **E** and **D** are respectively the vectors of strength and induction of an electric field, then the tensor of specific inductive capacitance, defined by the relations $D^i = \varepsilon^i_{\cdot k}E^k$, must be symmetric: $\varepsilon_{ki} = \varepsilon_{ik}$. From the derivation of these relations [20] it is clear that, in the nonlinear case, $\mathbf{D} = F(\mathbf{E})$, one must require that the function F [1] be a potential function, that is to say, that $D^i = \partial f(\mathbf{E}) / \partial E_i$, where $f(\mathbf{E})$ is a function of the invariants of the vector **E** with respect to the group G of symmetries of the medium.

The results obtained permit us to write down the form of the potential functions **V(B)** and **W(H)** (employing the notation of Sections 4 and 5):

groups $\infty / \infty\cdot m,\ \infty / \infty$

$$V^i = 2\,\frac{\partial f}{\partial\varphi}\, B^i \tag{6.1}$$

groups $m : \infty : m,\ \infty : 2,\ \infty : m$

$$V^i = 2\left(\frac{\partial f}{\partial\varphi_1}\,\zeta_\alpha^i + \frac{\partial f}{\partial\varphi_2}\,\gamma_\alpha^i\right) B^\alpha \tag{6.2}$$

groups $\infty\cdot m,\ \infty$

$$V^i = \frac{\partial f}{\partial\varphi_1}\,\zeta^i + 2\,\frac{\partial f}{\partial\varphi_2}\,\gamma_\alpha^i B^\alpha \tag{6.3}$$

groups $\infty / \infty\cdot m,\ \infty / \infty$

$$W^{ik} = 2\,\frac{\partial f}{\partial\varphi}\, H^{ik} \tag{6.4}$$

groups $m\cdot\infty : m,\ \infty : 2,\ \infty\cdot m$

$$W^{ik} = 2\left(\frac{\partial f}{\partial\varphi_1}\gamma^i_\alpha\gamma^k_\beta + \frac{\partial f}{\partial\varphi_2}\zeta^{[i}_\alpha\gamma^{k]}_\beta\right)H^{\alpha\beta} \tag{6.5}$$

groups $\infty : m,\ \infty$

$$W^{ik} = \frac{\partial f}{\partial\varphi_1}\Omega^{ik} + 2\frac{\partial f}{\partial\varphi_2}\zeta^{[i}_\alpha\gamma^{k]}_\beta H^{\alpha\beta} \tag{6.6}$$

In general the potential functions may be obtained also for other cases, for example

$$S^{ik} = \frac{\partial^2 f(\mathbf{B})}{\partial B_i \partial B_k}, \qquad W^{ik} = E^{ik}_{\cdot\cdot\alpha}\frac{\partial f(\mathbf{B})}{\partial B_\alpha}, \qquad V^i = E^i_{\cdot\alpha\beta}\frac{\partial f(\mathbf{H})}{\partial H_{\alpha\beta}} \tag{6.7}$$

where in Equations (6.7) it is supposed that the group G leaves invariant the geometrical tensor $\mathbf{E}$ (if this property does not hold, the corresponding functions are identically zero).

A comparison of the formulas for the potential functions with the general formulas of Sections 4 and 5 permits the determination of the conditions under which the tensor functions $\mathbf{V}(\mathbf{B})$ and $\mathbf{W}(\mathbf{H})$ are potential, conditions which vary from group to group under consideration. For example, let us look at the functions $\mathbf{V}(\mathbf{B})$:

Groups $\infty/\cdot\infty\cdot m$ and ∞/∞ : all functions $\mathbf{V}(\mathbf{B})$ are potential.

Groups $m\cdot\infty : m$ and $\infty : m$

$$\frac{\partial f_1}{\partial\varphi_2} = \frac{\partial f_2}{\partial\varphi_1} \tag{6.8}$$

Group $\infty : 2$

$$\frac{\partial f_1}{\partial\varphi_2} = \frac{\partial f_2}{\partial\varphi_1}, \qquad f_3 = 0 \tag{6.9}$$

Group $\infty\cdot m$

$$2\frac{\partial f_1}{\partial\varphi_2} = \frac{\partial f_2}{\partial\varphi_1} \tag{6.10}$$

Group ∞

$$2\frac{\partial f_1}{\partial\varphi_2} = \frac{\partial f_2}{\partial\varphi_1}, \qquad f_3 = 0 \tag{6.11}$$

In Sections 4 and 6 we gave all possible cases of the dependence of a polar and axial vectors, and also of the dependence of an arbitrary second rank tensor on a polar or axial vector and an arbitrary number of scalars, which are compatible with the symmetries of isotropic and nonisotropic media and different textures. Indeed, in the expansion $\dot{\mathbf{T}} = \Sigma f_{(\nu)}\mathbf{Q}_{(\nu)}$ the functions $f_{(\nu)}$ may depend not only on the invariants of the tensor $\mathbf{A}$, but also on an arbitrary number of other invariants of the group of symmetries of the medium, in particular, on scalars, that is, the invariants of the group $\infty/\infty\cdot m$. It is also clear that the formulas of Sections 4 and 6 also give the most general arbitrary second rank tensor having the specified properties (and not only just the symmetric or antisymmetric); because an arbitrary second rank tensor T^{ik} has the form

$$T^{ik}(\mathbf{A}) = S^{ik}(\mathbf{A}) + W^{ik}(\mathbf{A})$$

In a rectangular Cartesian system of coordinates, with base vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ (see the beginning of Section 4), all formulas given take a simple form. They take on an even simpler appearance when, making use of the freedom of rotation of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in their plane, one chooses the vector $\mathbf{e}_2$ perpendicular to the vector $\mathbf{B}$ or to the axial vector $\mathbf{H}$.

7. The fulfillment of the requirements discussed at the beginning of Section 3 enables one to obtain important information from the formulas of Sections 4 and 5. Indeed, these formulas permit one to obtain easily the general form of tensors of internal symmetry $[V^q]$, $V[V^q]$, $\{V^2\}[V^q]$, $[V^2][V^q]$, $V^2[V^q]$, $[\{V^2\}^q]$, $V[\{V^2\}^q]$, $\{V^2\}[\{V^2\}^q]$, $[V^2][\{V^2\}^q]$, $V^2[\{V^2\}^q]$, which are invariant with respect to the given Curie group, for any q .

Each of the tensors $\mathbf{T}_{(q)}$ in the expansion (3.1) is of the form

$$T_{(q)}^{i_1\ldots i_r} = P_{(q)}^{i_1\ldots i_r}{}_{\beta_1^{(1)}\ldots\beta_s^{(1)}\ldots\beta_1^{(q)}\ldots\beta_s^{(q)}} A^{\beta_1^{(1)}\ldots\beta_s^{(1)}}\ldots A^{\beta_1^{(q)}\ldots\beta_s^{(q)}} \tag{7.1}$$

The tensor $\mathbf{P}_{(q)}$ is defined by (7.1) up to an arbitrary permutation of the group of indices $\beta_1^{(l)}\ldots\beta_s^{(l)}$; it is therefore natural to seek to symmetrize with respect to all transformations of this group of indices. Under this condition, the tensor $\mathbf{P}_{(q)}$ is uniquely determined, and its internal symmetry equal $\tau\times[\alpha^q]$. The tensor $\mathbf{P}_{(q)}$, obviously, is invariant with respect to the group G which is defined by the geometrical tensors $\mathbf{K}_{(\pi)}$. The number of its linearly independent components is $m_q = m_q^*$ (see Section 3). Thus we may assert that, when the polynomial coefficients $f_{(\nu)}$ (of degree $q-c(\nu)$) are allowed to take on all admissible real values, then the tensors $\mathbf{P}_{(q)}$ vary over all possible linear spaces of tensors of internal symmetry $\tau\times[\alpha^q]$, with constant real components, which are invariant with respect to the group G .

Let us apply this method in the following simple case: let us determine the tensor of rank nine, $\mathbf{P}$, with internal symmetry $[V^2][V^7]$, which is invariant with respect to the group $\infty:2$ (see Section 4). In order to determine the tensors $\mathbf{S}_{(7)}$ (of seventh degree with respect to the components of $\mathbf{B}$), we must put

$$f_1 = f_2 = f_4 = f_5 = 0,\qquad f_3 = k_1\varphi_1^3 + k_2\varphi_1^2\varphi_2 + k_3\varphi_1\varphi_2^2 + k_4\varphi_2^3$$
$$f_6 = k_5\varphi_1^2 + k_6\varphi_1\varphi_2 + k_7\varphi_2^2$$

Substituting the expressions for φ_1 and φ_2, we get

$$S_{(7)}^{ij} = \zeta^{(i}\Omega^{j)}_{\cdot\beta}B^\beta\,(k_1\zeta_{\varkappa\lambda\mu\nu\rho\sigma} + k_2\zeta_{\varkappa\lambda\mu\nu}\gamma_{\rho\sigma} + k_3\zeta_{\varkappa\lambda}\gamma_{\mu\nu}\gamma_{\rho\sigma} + k_4\gamma_{\varkappa\lambda}\gamma_{\mu\nu}\gamma_{\rho\sigma})\,B^\varkappa B^\lambda B^\mu B^\nu B^\rho B^\sigma +$$
$$+ \zeta_\beta\Omega^{(i}_{\cdot\varkappa}\gamma_\lambda{}^{j)}B^\beta B^\varkappa B^\lambda\,(k_5\zeta_{\mu\nu\rho\sigma} + k_6\zeta_{\mu\nu}\gamma_{\rho\sigma} + k_7\gamma_{\mu\nu}\gamma_{\rho\sigma})\,B^\mu B^\nu B^\rho B^\sigma$$

The sought tensor $\mathbf{P}$, properly symmetrized by the set of the coefficients for $B^\beta\ldots B^\sigma$, that is

$$P^{ij}_{\cdot\cdot klmnprs} = k_1\Omega^{(i}_{\cdot(k}\zeta^{j)}_{lmnprs)} + k_2\Omega^{(i}_{\cdot(k}\zeta^{j)}_{lmnp}\gamma_{rs)} + k_3\Omega^{(i}_{\cdot(k}\zeta^{j)}_{lm}\gamma_{np}\gamma_{rs)} +$$
$$+ k_4\Omega^{(i}_{\cdot(k}\zeta^{j)}\gamma_{lm}\gamma_{np}\gamma_{rs)} + k_5\Omega^{(i}_{\cdot(k}\gamma^{j)}_{l}\zeta_{mnprs)} + k_6\Omega^{(i}_{\cdot(k}\gamma^{j)}_{l}\gamma_{mn}\zeta_{prs)} + k_7\,\Omega^{(i}_{\cdot(k}\gamma^{j)}_{l}\gamma_{mn}\gamma_{pr}\zeta_{s)}$$

Thus, we have solved here, almost without any calculations, for tensors of arbitrarily high order, the problem proposed and solved in [4 and 21] only for tensors of the four lowest ranks; however, the problem is solved here only for tensors which possess the specified internal symmetries.

8. The Hamilton-Cayley formula

$$T^{ik} = f_1 g^{ik} + f_2 A^{ik} + f_3 A^i_{\cdot\alpha}A^{\alpha k},\qquad f_{1,2,3} = f_{1,2,3}\,(A^\alpha_{\cdot\alpha},\ A^\alpha_{\cdot\beta}A^\beta_{\cdot\alpha},\ A^\alpha_{\cdot\beta}A^\beta_{\cdot\gamma}A^\gamma_{\cdot\alpha}) \tag{8.1}$$

gives the general functional dependence of a single symmetric tensor of the second rank on another which is compatible with the isotropy of the medium. This formula plays an important role in the nonlinear mechanics of an isotropic continuous medium [1]; from it, in particular, one derives much more special relations which are employed in the nonlinear theory of elasticity (*). An analogous role, in the nonlinear theory of isotropic continuous media, is played by the more obvious relations of the form

$$V^i = f(B_\alpha B^\alpha) B^i, \qquad S^{ik} = f_1 (B_\alpha B^\alpha) g^{ik} + f_2 (B_\alpha B^\alpha) B^i B^k \tag{8.2}$$

Generally, formulas of the type $\mathbf{T} = F(\mathbf{A})$, which represent the general form of the functional dependence of a tensor $\mathbf{T}$ on a tensor $\mathbf{A}$ (or , in a still more general case, on tensors $\mathbf{A}_1, \ldots, \mathbf{A}_{(m)}$), which is compatible with an isotropic medium, may be referred to as the generalized Hamilton-Cayley formula for isotropic media.

The formulas of Section 4, therefore, may be called the generalized Hamilton-Cayley formulas which represent the dependence of a vector, a symmetric tensor. an antisymmetric tensor, and a general tensor of the second rank, on a vector, for a medium with limit symmetry. The corresponding formulas of Section 5 may be regarded as the generalized Hamilton-Cayley formulas which represent the dependence of the quantities in question on an axial vector, or on an antisymmetric tensor of the second rank.

Perhaps the Hamilton-Cayley formulas for potential dependence should also be regarded as an important particular case. Some of these formulas are given in Section 6.

The subject and the fundamental ideas of this paper arose from discussions with L.I. Sedov. The author wishes to express his deepest thanks to L.I.Sedov and V.V.Lokhin.

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*) See, for example, [2], Chapter II, Section 15 and Chapter III, Section 15, and also [3], Chapter IV, Section 2.

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Translated by J.B.D.